math 251 - week 9 - Ch 5 **Eigen Values & Eigen Vectors**

def. $|\lambda I - A| = 0 \Rightarrow$ Characteristic Equation.

the root of the characteristic equation is called **Eigen Values**.

Corresponding to each eigen value there is a **Eigen Vector**. $|\lambda I - A| x = 0$

Ex Find the eigen value of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

Sol. Write the characteristic equation is

$$|\lambda I - A| = 0 \Rightarrow = \begin{vmatrix} \lambda - 3 & 0 \\ 8 & \lambda - (-1) \end{vmatrix} = 0$$

$$\Rightarrow = (\lambda - 3)(\lambda + 1) - (-8)(0) = 0$$

$$\Rightarrow (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 3 \text{ or } \lambda_2 = -1 \text{ are the}$$

Eigen Values of A.

Ex Find the eigen value of $A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -1/9 \end{bmatrix}$

Sol. the eigen values of A is

$$\lambda_1 = 1/2 \ , \ \lambda_2 = 2/3 \ , \ \lambda_3 = -1/9$$

Ex) Determine the eigen value and corresponding Eigen Vectors. $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$

Sol. the characteristic equation is

$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-3 & -1 \\ -6 & \lambda-2 \end{vmatrix} = 0$

$\Rightarrow (\lambda-3)(\lambda-2) - (-6)(-1) = 0$

$\Rightarrow \lambda^2 - 5\lambda = 0 \rightarrow \lambda(\lambda-5) = 0$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 = 5$ are the eigen values.

Eigen Vector corresponding to $\lambda_1 = 0$

$(\lambda I - A)x = 0$ Put $\lambda_1 = 0$ in equation :-

$\Rightarrow \begin{bmatrix} 0-3 & -1 \\ -6 & 0-2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & -1 \\ -6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\xrightarrow{-2R_1+R_2} \begin{pmatrix} -3 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Since, the rank of this matrix is One

(No. of nonzero rows)

x_1 is leading variable and x_2 is free Variable

$-3x_1 - 1x_2 = 0 \Rightarrow -3x_1 = x_2$

Let (Assume) $x_2 = 3 \Rightarrow$ then $x_1 = -1$

So $(-1, 3)$ is a Eigen vector.

Assume $x_2 = 6$ then $x_1 = -2 \Rightarrow (-2, 6)$ is a Eigen vecter.

(infinte no. of eigen vectors) for $\lambda_1 = 0$

Eigen vector corresponding to $\underline{\lambda_2 = 5}$

$$\begin{pmatrix} 5-3 & -1 \\ -6 & 5-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -1 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \xrightarrow{3R_1 + R_2} \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since, the rank is One.

x_1 is the leading variable & x_2 is free variable.

$\Rightarrow 2x_1 - x_2 = 0 \Rightarrow \underline{2x_1 = x_2}$

let $x_2 = 2$, then $x_1 = 1$ $\quad (1,2)$ & $(2,4)$ the eigen vectors.

let $x_2 = 4$, then $x_1 = 2$

Ex) find the eigen value of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Sol. the characteristic equation is

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-0 & -1 & -0 \\ -0 & \lambda-0 & -1 \\ -4 & +17 & \lambda-8 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{vmatrix} = 0$$

$$\Rightarrow \lambda \begin{vmatrix} \lambda & -1 \\ 17 & \lambda-8 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 \\ -4 & \lambda-8 \end{vmatrix} + 0 \begin{vmatrix} 0 & \lambda \\ -4 & 17 \end{vmatrix} = 0$$

$$\Rightarrow \lambda [\lambda^2 - 8\lambda + 17] + (0-4) + 0 = 0$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \Rightarrow \underline{3 \text{ roots}}$$

hint: $\lambda = \pm 1, \pm 2, \pm 3, \pm 4$

$\lambda=1 \Rightarrow 1^3 - 8 \times 1^2 + 17 . 1 - 4 = 0 \Rightarrow 6 \neq 0$ Not satisfay

$\lambda=4 \Rightarrow 4^3 - 8 \times 4^2 + 17 . 4 - 4 = 0 \Rightarrow 0 = 0$ Satisfy

so, $(\lambda - 4)$ is one factor of $\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$

divide

$$\begin{array}{r} \lambda^2 - 4\lambda + 1 \\ \lambda - 4 \,\big)\, \lambda^3 - 8\lambda^2 + 17\lambda - 4 \\ -\;\; \lambda^3 \mp 4\lambda^2 \qquad\qquad\quad \\ \hline -4\lambda^2 + 17\lambda \quad\;\; \\ \mp 4\lambda^2 \pm 16\lambda \quad\;\; \\ \hline 1\lambda - 4 \\ -\; 1\lambda \mp 4 \\ \hline 0 \end{array}$$

$$(\lambda - 4)\underbrace{(\lambda^2 - 4\lambda + 1)}_{\text{Solve it by}} = 0$$

Solve it by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\lambda_1 = 4$, $\lambda_2 = 2 + \sqrt{3}$, $\lambda_3 = 2 - \sqrt{3}$

is the eigen values.

EX/ find the eigen value & Eigen vector & Diagonalize the matrix.

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Sol. The characteristis equation

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-0 & -0 & +2 \\ -1 & \lambda-2 & -1 \\ -1 & -0 & \lambda-3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$\underline{\underline{\lambda=1}}$ is satisfy $\Rightarrow$ $(\lambda-1)$ is one root.

$$\Rightarrow (\lambda-1)(\lambda-2)^2 = 0$$

$\Rightarrow$ $\lambda = 1, 2, 2$ are the E. values.

Eigen vector corresspondling to $\underline{\lambda=1}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow[R_1+R_3]{R_1+R_2}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

* the rank of the matrix is $\underline{\underline{2}}$.

The Leading Variable x_1 and x_2 and the free variable is x_3.

$$\left.\begin{aligned} x_1 + 0x_2 + 2x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= 0 \end{aligned}\right\}$$

$\Rightarrow \boxed{x_1 = -2x_3}$ and $\boxed{x_3 = x_2}$

Let $\underline{x_3 = 2}$, then $\underline{x_2 = 2}$ and $\underline{x_1 = -4}$

the eigen vector $\begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}$

The eigen vectors for $\underline{\lambda = 2}$ is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$P = \begin{bmatrix} -4 & -1 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$